On the epistemological status of mathematical objects in Plato's philosophical system

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Abstract

The main question to be discussed in this paper is what the epistemological status of mathematical objects is in Plato's thought system, as opposed to that of objects from the sensible realm. After a brief introduction, some intuitions about mathematics that can be established by mere reflection on its procedures will be discussed, and it will be established that mathematics proceeds from axioms via previously defined rules to conclusions that depend on the choice of principles. It also becomes clear that there is the desire for an answer to the question how there can be a conception of mathematical objects in our understanding if we have never experienced them as such in the visible world. Next will be dealt with the fundamentals of Plato's epistemological theory of forms, in order to be able to find the place for mathematical objects. It will be shown consequently that Plato's conception of mathematics can be equated with the intuitions that are established before. In the last section there will be made combinations of elements of the previous sections into an answer to the main question.

1 Introduction

It has been reported that above the door of the Academy in Athens, which was founded by Plato, the following words were written: *Let no one unversed* *in geometry enter here*. Furthermore, the dialogue *Timaeus* contains an elaboration on Plato's theory of the constituents of the universe that make up all of the tangible world – Plato thought of these as regular mathematical shapes.

It is then evident that Plato thinks highly about mathematics. This is the reason that examples from geometry are found throughout his dialogues.

Nevertheless there remains an essential question as to their nature –; what are mathematical objects, how can we know them – if we can, and what is their epistemological status in comparison with tangible objects on one hand and Platonic forms on the other? These questions will then lie at the fundaments of the discussion in this paper.

1.1 Problem

The most fundamental problem to be addressed in this paper can be formulated as follows:

> What is the epistemological status of mathematical objects in Plato's philosophical system, as opposed to that of objects of the sensible realm?

1.2 Outline

The remainder of this paper is organised as follows. In section 2 there will be made an attempt to provide an account of the intuitions about mathematical procedures. This will be arrived at by using *The Elements of Geometry* by Euclid as an example. In section 3 there will be a detailed study of sections of Book V, VI and VII of *The Republic* by Plato, and some extracts from *Meno*. On the basis of this there will be a discussion on to what extent Plato's conception of mathematical objects is compatible with the intuitions described before in section 4. In section 5 there will be an effort to make combinations of what is established in the previous sections in order to come to a formulation of an answer to the main question.

2 Intuitions on mathematical procedures

In this section, some of the intuitions about mathematics will be made explicit, such that it will be possible to interpret Plato's thoughts on the philosophy of mathematics on the basis of these intuitions –, and to see that neither one excludes the other.

The workings of mathematics will be discussed using *The Elements of Geometry* by Euclid as an example, since this work is considered to form the fundaments of western mathematics.

2.1 Definitions

Euclid starts out *Book I* of *The Elements of Geometry* as follows:

> Definition 1. A point is that which has no part.
>
> Definition 2. A line is breadthless length.
>
> Definition 3. The ends of a line are points.

The first definition asserts that a point is something that has no length, breadth, height, or any other spatial dimension. The second asserts that a line, though with length bestowed, has no breadth or height or any other thinkable spatial dimension. The third asserts that once a line has one or more ends, these ends are distinct points.

Now several things can be established on the basis of this.

First of all, this is the very starting point of the geometrical work. In other words, these definitions are assumed and, as such, not proven.

- It is not immediately apparent that it is in principle at all – or at all not – possible to prove these assertions, either within the framework of the mathematical science or outside it – by another branche of cognition. It is perceivable that these definitions can only be proven on the basis of others, who would then, in turn, require proof. In the case of Euclid's *Elements* there is made no attempt at all to justify the definitions and therefore it seems valid to assume that Euclid assumed that these definitions did not require proof in order for his mathematical theory to still be useful and meaningful.

- The definitions can be seen as arbitrary in the sense that – for instance – the concept of something having no part could be given another name and that which is called a line could be defined as referring to another concept.

- The definitions are very explicit –; Euclid overtly tries to evade making assumptions not explicit.

These observations will be combined to form the concept of *axiom*, as it will be used in this paper.

Furthermore, it is fruitful to note the fact that something that has no extension, such as a point, or something that has only extension to a certain degree, such as a line, could never have been present to our senses in the visible or sensible world. Nevertheless, there can be a conception of some sort or another in our mind of this which is called a point. One of the fundamental questions that will have to be addressed by a philosopher in order for him to incorporate mathematical objects in his ontological or epistemological system, is how this can be – that we, even though we have never experienced mathematical objects as such among the visible things, there can be a conception of them in our minds. From now on this property that is shared by many, if not all, mathematical objects will be referred to as *abstractness*.

2.2 Postulates

After introducing many additional definitions, Euclid continues:

> Postulate 1. To draw a straight line from any point to any point.

Here it is asserted that if there are any two points, there can be a straight line that connects them.

It is noteworthy that postulates bear an intimate relationship with definitions in the sense that this postulate – and likewise generally any other – elaborates on the terms that were defined before in the sense that a property of the relationship between two previously defined entities is established. In this way, it can be argued that this postulate provides more meaning to the concepts of line and point.

2.3 Common notions

After several more postulates, similar in nature to the one seen before, Euclid introduces "common notions":

> Common notion 1. Things which equal the same thing also equal one another.

> Common notion 2. If equals are added to equals, then the wholes are equal.

It can, again, be established that definitions, postulates and common notions are very much alike in the sense that none of them is proven in any way. Either of them is therefore by nature axiomatic. "Definitions" are then used to define the objects of mathematical inquiry, the "postulates" are drawn in to define relations between them, and the "common notions" serve to clear the way for mathematical reasoning – which will proceeds by *deduction*. It is also noteworthy that the term "common notions" reflects the fact that these axiomatic principles are not proven, but hoped to be agreed upon.

2.4 Propositions

Next, Euclid poses the following proposition:

> Proposition 1. To construct an equilateral triangle on a given finite straight line.

This assertion is accompanied by a drawing which illustrates graphically the way in which the equilateral triangle can be constructed if a finite straight line is given. If this line has endpoints A and B (which are given), then two circles are constructed; one having A in its center and AB as a radius, the other centered at B, having BA as its radius. Then the equilateral triangle consists of the points A, B, and the third point is one of the points of intersection of the two circles.

The purpose of this discussion is not to provide a rigorous mathematical proof (especially since not all of Euclid's definitions, postulates and common notions are given), but merely to understand the nature of the reasoning that lead up to this point. First of all there are defined abstract entities, the relations between them, some properties and common notions about how these should be derived. Following from this was a postulate.

This postulate is then fundamentally different from the principles that have been established in the sense that the principles could not be proven. This postulate can, and about it the following can be established:

- It is general: the theorem pertains not only to the particular line, points and circles that we had in mind when visualising the procedure, but it pertains to any line, point and circle – properly understood by the definitions given for them – that can be imagined. It will be held here that this is the same as saying that the postulate includes abstract entities only. As such, it establishes nothing about the visible world.

- It is solely based on definitions, postulates, common notions, i.e. on axioms. It can be said that the derivation has proceeded according to the rules made explicit among, for instance, the common notions. At any point in the derivation, there was a rule by which a the following step could be taken. This could lead up to the conclusion that once there is agreement about the acceptability of the axioms, there can be no disagreement about the acceptability of the conclusions – and although this latter point is not at all evident, there will be given an example the other way around; if there is disagreement about one of the axioms, then there will, generally, be disagreement about the conclusion, too.

As to this latter point, it is interesting to see that so far, mathematics has proceeded purely *a priori*. In order to arrive at the conclusion – that a equilateral triangle can be constructed on the basis of a given line by means of the steps described above – there has been no survey of any number of examples of lines and points and measuring the result, but rather a meditation on the *concepts* of lines and points.

A final note will close this section – since mathematics proceeds in this *a priori* way, it can potentially be applied to reality, but validity of its conclusions will only extend to the sensible objects precisely in so far as they share the relevant properties of the defined objects on the basis of which the derivation has been made.

This is reflected in application of mathematics to sensible reality. The typical procedure can be sketched as follows:

1. We establish to which extent the sensible objects share the relevant properties of the mathematical objects.

2. We see what relationships between the mathematical objects are known.

3. We conclude that these relationships, which hold for the mathematical objects, must hold for the visible objects too.

It is interesting to contrast this with the following procedure:

1. We see the visible objects.

2. We derive the (mathematical) relationship between them on the basis of *these particular objects*.

3. We conclude that a certain relationship holds between.

The reason for the fact that the former procedure occurs more natural lies in the fact that mathematical conclusions are general, and therefore a derivation made once will not have to be repeated. Furthermore, it can be argued that the resemblances between visible objets and abstract mathematical objects are only limited; in mathematics there can be a straight angle which is precisely straight, all angles in the visible only approach straightness.

2.5 Summary of intuitions on mathematics

In summary, mathematics – in our conception of it – proceeds from axioms (definitions of abstract entities, postulates and common notions), to propositions by means of derivations that are (1) defined in advance, and (2) do not depend on sensory experience in any way, except by means of examples.

A question that arises immediately is how there can be a conception of mathematical objects in our understanding if we have never experienced them as such in the visible world.

3 Fundamentals of Plato's epistemology

3.1 Knowledge

In *The Republic*, the work that will form the basis of this survey, Plato starts the sections on epistemology and ontology in books VI and VII out arguing that those he calls "philosophical natures," should be seeking to understand precisely those things that are eternal and unchanging [485b]:

> Let's agree that philosophic natures always love the sort of learning that makes clear to them some feature of the being that always is and does not wander around between coming to be and decaying.

It follows from this assertion that there must be another realm of things which are not subject to decay as the visible and sensible things that are ordinarily present to our senses, and that this realm is accessible to human intellect in some way. These unchanging entities are not present in the sensible world and they are the forms.

Furthermore, on the basis of this there can be made a distinction between knowledge, opinion, and ignorance. According to Plato, knowledge is that which pertains to that which is, whereas ignorance pertains to that which is not, and opinion is in between them and pertains to that which cannot be said to be in itself, but cannot be said not to be either – and this is the sensible world.

and [479c]:

> As for those who study the many beautiful things, but do not see the beautiful itself [...] –these people, we shall say, opine everything but have no knowledge of anything they opine.
>
> Necessarily.
>
> What about the ones who in each case study the things that are always the same in every respect? Won't we say that they know and don't opine?
>
> That's necessary too.

It is apparent that knowledge must correspond to *that which is in itself*, such as Plato mentions, by example, "the beautiful itself." Such a thing, then, must be among the forms. The form is the one in which everything that makes a thing what it is is united. In this sense a form can be viewed as an essence or a principle, or, as Plato calls it, the *"being"* of something [507b]:

> And beauty itself and good itself and all the things that we thereby set down as many, reversing ourselves, we set down according to a single form of each, believing that there is but one, and call it "the being" of each.

In this way there can be accounted for the apparent variation that is experienced in this visible world - all slightly different things of one kind are united in their unique form.

On the basis of this distinction in epistemological status of objects in the visible world and in the world of forms there can also be different modes of understanding -; the understanding pertaining to the visible world and the understanding pertaining to the world of the forms. This distinction is made explicit by Plato [507b]:

> And we say that the many beautiful things and the rest are visible but not intelligible, while the forms are intelligible but not visible.

This, then, leads to the conception of epistemologically differing realms of objects and, correspondingly, of understanding.

3.2 Realms

Plato compares the perception of everyday objects in the visible realm with the (mental) cognition of the forms – in a sense the word "perception" could be applied to both. However, Plato makes one further nuance in the picture [507d]:

> Sight may be present in the eyes, and the one who has it may try to use it, and colors may be present in things, but unless a third kind of thing is present, which is naturally adapted for this very purpose, you know that sight will see nothing, and the colors will remain unseen.
>
> What kind of thing do you mean?
>
> I mean what you call light.

In this passage, Plato implicitly refers to the analogy between perception in the visible realm and cognition in the intelligible realm. In the way light makes perception in the visible realm possible, it is the form of good that is at the top of the hierarchy of the forms and that makes cognition in the intelligible realm possible [509b]:

> Therefore, you should also say that not only do the objects of knowledge owe their being known to the good, but their being is also due to it, although the good is not being, but superior to it in rank and power.

Plato now asserts that all things be divided in things intelligible and things visible. Among the things visible, there can be a distinction between what Plato calls "images" of things and the originals of them [509d]:

> And by images I mean, first, shadows, then reflections in water and in all close-packed, smooth, and shiny materials, and everything of that sort, if you understand.

4 Intelligible Realm

4.1 General distinction

The division of the intelligible realm brings closer the core issue discussed in this paper. Plato asserts, by calling in the analogy of the division of a line in unequal section, the concept that in the intelligible realm there can be made a similar distinction as in the sensible realm [510b]:

> In one subsection, the soul, using as images the things that were imitated before, is forced to investigate from hypotheses, proceeding not to a first principle but to a conclusion. In the other subsection, however, it makes its way to a first principle that is *not* a hypothesis, proceeding from a hypothesis but without the images used int he previous subsection, using forms themselves and making its investigation through them.

4.2 Hypotheses

In order to gain a clear understanding of this, it is useful to refer to the dialogue *Meno* for clarification of the concept of hypothesis [87]:

> if [geometers] are asked whether a specific area can be inscribed in the form of a triangle within a given circle, one of them might say: "I do not yet know whether that area has that property, but I think I have, as it were, a hypothesis that is of use for the problem, namely this: If that area is such that when one has applied it as a rectangle to the given straight line in the circle it is deficient by a figure similar to the very figure which is applied, then I think one alternative results, whereas another results if it is impossible for this to happen.

First of all, it can be noted that this hypothesis is either the case or not – there cannot be a further nuance –, and employing it simplifies the mathematical complexity of the problem into two complementary situations that are equally perceivable.

This leads to the inevitable observation that the geometer has not provided the one who asked him with a definite answer to the question in the form of a statement of what must be the case. Rather, he formulates a series of conditional ("if A_n then B_n"-)assertions (of which there can generally be given any particular number, but in this case there will be two, since the two cases are complementary), where A_n is one hypothesis and B_n that which follows from it. It is then left to the person who asked the question to decide which of the hypotheses A_n is the case, and then see which factual consequence B_n follows.

This illustrates the essence of a *conclusion* and it is distinguished by Plato from a *first principle*, in the sense the former is conditional and the latter is not and pertains to the realm of unchanging objects, i.e. the forms.

4.3 Mathematics

In this way Plato explains that mathematics will only be capable of providing answers that rely on hypotheses, leaving it to others to decide on whether this hypothesis is true. This is also seen in the analysis of Euclids *Elements*, where the propositions will only be true if (1) the axioms involved are accepted, and (2) the proposition is correctly applied. An example of incorrect application of the proposition that served as an example would be to take a circle instead of a *straight* line. In this case the procedure described by the proposition will not yield a equilateral triangle.

Furthermore, Plato considers the use of visible figures in mathematics (such as drawing the mathematical objects) as a means to direct thought towards the abstract entities they resemble. Plato refers to these abstract entities as "the things in themselves" – and that reveals there form-like nature [510d]:

> [a]lthough [geometers] use visible figures and make their claims about them, their thought isn't directed to them but to those other things that they are like. They make their claims for the sake of square itself and the diagonal itself, not the [particular] diagonal they draw, and similarly with the others.

In *Phaedo* there is a discussion about equality, which, as we saw above, played an important role in the mathematical derivation (for instance, in the assertion that of an equilateral triangle, the sides are all *equal*). Plato argues that of equality there is a form and everything in the visible realm that participates in this form and is present to us merely approaches this form and can never *be* the form:

> But what would you say of equal portions of wood and stone, or other material equals? and what is the impression produced by them? Are they equals in the same sense as absolute equality? or do they fall short of this in a measure?
>
> Yes, he said, in a very great measure, too.

This, then, is again in line with the properties of mathematical objects established in section 2.

The axiomatic nature of the first principles is also touched on by Plato [510c]:

> They make these their hypotheses and don't think it necessary to give any account of them, either to themselves or to others, as if they were clear to everyone.

The reason why the mathematician called in in the *Meno* is unable to answer the question is the lack of certainty about the hypothesis. For this reason, Plato feels that geometry has no workings outside the sphere of influence of its axioms and therefore it cannot be genuinely informative about fundamental truths [511]:

> This, then, is the kind of thing that, on the one hand, I said is intelligible, and, on the other, is such that the soul is forced to use hypotheses in the investigation of it, not travelling up to a first principle, since it cannot reach beyond its hypotheses.

4.4 Understanding in the intelligible realm

In this way mathematical thought must be distinguished from direct inquiry into the forms, which might legitimately use hypotheses initially (Socrates does just that in the *Meno*), but will eventually push away the ladder of hypotheses, to reach genuine certainty that does not depend on hypotheses [513b]:

> It does not consider these hypotheses as first principles but truly as hypotheses – but as stepping stones to take off from, enabling it to reach the unhypothetical first principle of everything.

Furthermore, this genuinely fundamental inquiry into the forms does not make use of anything sensible like the geometrical diagrams, being imperfect images representing the prefect mathematical forms – but only of the forms [513b]:

> Having grasped this principle, it reverses itself and, keeping hold of what follows from it, comes down to a conclusion without making use of anything visible at all, but only of forms themselves, moving on from forms to forms, and ending in forms.

5 Combinations

5.1 Interpretation of the realms

Inhibiting our interpretation of to what objects the intelligible realm corresponds is the fact that Plato in *The Republic* elaborates on the *objects* of knowledge in the visible realm, but only explicitly the modes of knowing corresponding to the intelligible realm. For instance, Plato introduces tangible objects and their images as corresponding to the visible realm, but as to the intelligible realm, Plato provides an account of the different ways of cognition of the soul.

5.2 Conception of Mathematical Objects

The question that arises when discussing intuitions about mathematical procedures is how there can be a conception of mathematical objects in our understanding if we have never experienced them as such in the visible world.

By assuming the form-like nature of mathematical objects Plato has reduced this question to the question that is more general and which lies at the fundament of the *Meno* dialogue, and is asked by Meno himself [80d]:

> How will you look for it, Socrates, when you do not know at all what it is? How will you aim to search for something you do not know at all? If you should meet with it, how will you know that this is the thing that you did not know?

The answer to this question would deserve even more than a paper of its own – very briefly put, it is that "learning is recollection" [81e]; the souls know the forms when they are seperate from a body, and learning is recollecting that which has already been present to the soul.

As for mathematical objects, it can be argued in line with Plato, that lines, circles and points in the visible everyday world remind us of the eternal forms for line, circle and point that have been present to our soul before birth.

6 Answer to the problem

Therefore, it is argued in this paper that Plato considers mathematical entities forms, even though mathematical inquiry into them is different than inquiry into forms as it is done by philosophers.

Mathematical entities are form-like, because they are "things in themselves", or essences; and things we find in reality can only approach them because they are ontologically secondary, and forms are primary.

Furthermore, mathematical diagrams that help to visualise problems to geometers, but they are not the objects of their inquiry – only forms are that, and the diagrams are only representations of them.

7 References

All references correspond to the following English translations but original page numbering, reprinted in *Classics of Western Philosophy*, 5th ed., edited by Steven M. Cahn, Hackett Publishing Company, 1999:

- Plato, *Meno*, translated by G.M.A. Grube, Indianapolis: Hackett Publishing Company, 1976.
- Plato, *Phaedo*, translated by G.M.A. Grube, Indianapolis: Hackett Publishing Company, 1977.
- Plato, *Republic*, translated by G.M.A. Grube, revised by C.D.C. Reeve, Indianapolis: Hackett Publishing Company, 1992.